Schur multiple Eisenstein series

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Abstract

Eisenstein series provide the classical examples of modular forms, with their constant Fourier coefficients given by Riemann zeta values. In 2006, Gangl, Kaneko, and Zagier introduced double Eisenstein series to relate double zeta values to modular forms, and Bachmann (2012) later generalized this to multiple Eisenstein series. This survey article explores a further generalization, called Schur multiple Eisenstein series. These series have Fourier expansions whose constant terms are Schur multiple zeta values and satisfy certain relations similar to Schur multiple zeta values, such as the Jacobi-Trudi formula. We also consider some special cases related to modularity. Further we introduce a certain Hopf algebra which can be used to describe the Fourier expansion of Schur multiple Eisenstein series.

1 Introduction

This report summarizes some of the results obtained by the author in [Yu], where Schur multiple Eisenstein series are introduced. These generalize classical Eisenstein series which are examples of modular form for the full modular group. For an even integer $k > 2$, and an element τ in the complex upper half plane $\mathbb{H}$, the Eisenstein series of weight k is defined by

$$G(k;\tau) = \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k} = \zeta(k) + \frac{(2\pi i)^k}{(k-1!)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \quad q = e^{2\pi i\tau}.$$

One important result in the theory of modular forms is that Eisenstein series are the fundamental building blocks of all modular forms. Specifically, every modular form can be expressed as a polynomial in $G(4;\tau)$ and $G(6;\tau)$. The second expression is its Fourier series, the coefficients are divisor function $\sigma_k(n) = \sum_{d|n} d^k$, and the constant term is the Riemann zeta function $\zeta(k) = \sum_{m=1}^{\infty} \frac{1}{m^k}$.

1.1 Multiple zeta values and multiple Eisenstein series

The product of two Riemann zeta values can be expressed as

$$\begin{aligned} \zeta(r)\zeta(s) &= \zeta(r,s) + \zeta(s,r) + \zeta(s+r) \quad (r,s\geq 2), \\ \zeta(r)\zeta(s) &= \sum_{j=2}^{s+r-1} \left[\binom{j-1}{r-1} + \binom{j-1}{s-1} \right] \zeta(j, s+r-j) \quad (2 \leq j \leq \frac{s+r}{2}), \end{aligned} \tag{1.1}$$

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where $\zeta(r,s)=\sum_{0<m<n}\frac{1}{m^rn^s}$ are called double zeta values, which are known to satisfy a collection of relations. The relations (1.1) are double shuffle relations. In particular, the first expression for the product is referred to as the stuffle relation, while the second one is called shuffle relation. Since the Riemann zeta functions are the constant term of Eisenstein series, the natural question arises: "Do Eisenstein series also satisfy these relations derived from zeta values?" In 2006 Gangl, Kaneko and Zagier [GKZ] showed that the structure of the $\mathbb{Q}$-vector space of all relations among double zeta values of weight k is connected in many different ways to the structure of the space of modular forms M_k of weight k on the full modular group $\Gamma_1 = PSL(2,\mathbb{Z})$, by introducing double Eisenstein series. In 2012, Bachmann [Ba] generalized this idea to multiple version. He studied the multiple Eisenstein series which are defined by

$$G(s_1,\dots,s_r;\tau)=\sum_{\substack{0\prec\lambda_1\prec\cdots\prec\lambda_r\\ \lambda_i\in\mathbb{Z}\tau+\mathbb{Z}}}\frac{1}{\lambda_1^{s_1}\cdots\lambda_r^{s_r}}\quad(\tau\in\mathbb{H};s_i\in\mathbb{N}_{\geq 2},\ i=1,\dots,r),$$

where the order $m_1\tau+n_1\prec m_2\tau+n_2$ means $m_1<m_2\vee(m_1=m_2\wedge n_1<n_2)$. Although not all multiple Eisenstein series are modular forms, we observe that $G(s_1,\dots,s_r;\tau)=G(s_1,\dots,s_r;\tau+1)$, which provides the Fourier expansion of multiple Eisenstein series.

Theorem 1.1 ([Ba]). *For $k_1,\dots,k_r\geq 2$ there exist explicit $\alpha^{k_1,\dots k_r}_{l_1\dots l_r,j}\in\mathbb{Z}$ such that*

$$G(k_1,\dots,k_r)=\zeta(k_1,\dots,k_r)+\sum_{\substack{0<j<r\\ l_1+\cdots+l_r=k_1+\cdots+k_r\\ l_1,\dots,l_r\geq 1}}\alpha^{k_1,\dots k_r}_{l_1\dots l_r,j}\zeta(l_1,\dots,l_j)g(l_{j+1},\dots,l_r)+g(k_1,\dots,k_r).$$

Here, in the case of Eisenstein series, $g(k)$ represents the generating series of the divisor sums. For higher depths, multiple versions are defined for $k_1,\dots,k_r\geq 1$ as follows:

$$g(k_1,\dots,k_r;\tau)=g(k_1,\dots,k_r)=(-2\pi i)^{k_1+\cdots+k_r}\sum_{\substack{0<m_1<\cdots<m_r\\ n_1,\dots,n_r>0}}\frac{n_1^{k_1-1}}{(k_1-1)!}\cdots\frac{n_r^{k_r-1}}{(k_r-1)!}q^{m_1n_1+\cdots+m_rn_r},$$

where $q=e^{2\pi i\tau}$. By the Lipschitz formula, the monotangent function $\Psi(k;\tau)$ is defined as

$$\Psi(k;\tau)=\sum_{n\in\mathbb{Z}}\frac{1}{(\tau+n)^k}=\frac{(-2\pi i)^k}{(k-1)!}\sum_{d>0}d^{k-1}q^d.$$

In particular, we can replace τ by $m_1\tau,\dots,m_r\tau$ and take the sum over all $1\leq m_1<\cdots<m_r$ to obtain

$$g(k_1,\dots,k_r)=\sum_{0<m_1<\cdots<m_r}\Psi(k_1;m_1\tau)\cdots\Psi(k_r;m_r\tau).\tag{1.2}$$

Also, the constant term corresponds to the multiple zeta values with same index, denoted by $\zeta(s_1,\dots,s_r)$. For integers $s_1,\dots,s_{r-1}\geq 1$ and $s_r\geq 2$, the multiple zeta values (MZVs) and multiple zeta-star values (MZSVs) are defined by

$$\zeta(s_1,\dots,s_r)=\sum_{0<n_1<\cdots<n_r}\frac{1}{n_1^{s_1}\cdots n_r^{s_r}},\quad\zeta^\star(s_1,\dots,s_r)=\sum_{0<n_1\leq\cdots\leq n_r}\frac{1}{n_1^{s_1}\cdots n_r^{s_r}}.$$

Since each term in Fourier expansion of multiple Eisenstein series contains an MZV factor, certain relations in the $\mathbb{Q}$-vector space of all MZVs can be extended to the $\mathbb{Q}$-vector space of all multiple Eisenstein series, such as the double shuffle relation (1.1).

1.2 Algebra setup

To study the double shuffle relations(1.1), Hoffman introduced the quasi-shuffle products [Ho]. Let L be an alphabet. A monic monomial in the non-commutative polynomial ring $\mathbb{Q}\langle L\rangle$ is called a word, and the empty word is denoted by $\mathbf{1}$. The symbol $\diamond$ represents a commutative and associative product on the $\mathbb{Q}$-vector space generated by L. The *quasi-shuffle product* $*_\diamond$ on $\mathbb{Q}\langle L\rangle$ is defined as a $\mathbb{Q}$-bilinear product satisfying $\mathbf{1} *_\diamond \mathbf{w} = \mathbf{w} *_\diamond \mathbf{1} = \mathbf{w}$ for any word $w \in \mathbb{Q}\langle L\rangle$ and

$$aw *_\diamond bv = a(w *_\diamond bv) + b(aw *_\diamond v) + (a \diamond b)(w *_\diamond v) \tag{1.3}$$

for any $a, b \in L$ and words $w, v \in \mathbb{Q}\langle L\rangle$. This defines a commutative $\mathbb{Q}$-algebra $(\mathbb{Q}\langle L\rangle, *_\diamond)$, which is called *quasi-shuffle algebra*. There have two different alphabets: $L_{xy} = \{x, y\}$, with the product $a \diamond b = 0$ for $a, b \in L_{xy}$ and $L_z = \{z_k | k \geq 1\}$, with the product $z_{k_1} \diamond z_{k_2} = z_{k_1+k_2}$ for $k_1, k_2 \geq 1$. The corresponding quasi-shuffle product of L_{xy} is $\sqcup\!\sqcup = *_\diamond$ referred to as the *shuffle product*, and the corresponding quasi-shuffle product of L_z is $* = *_\diamond$, referred to as the *stuffle product*.

Let $\mathcal{Z}$ be the $\mathbb{Q}$-vector space generated by all MZVs, and let $\mathfrak{H} = \mathbb{Q}\langle L_{xy}\rangle = \mathbb{Q}\langle x, y\rangle$. Define the subspaces of $\mathfrak{H}$ as $\mathfrak{H}^0 = \mathbb{Q} + x\mathfrak{H}y \subset \mathfrak{H}^1 = \mathbb{Q} + \mathfrak{H}y \subset \mathfrak{H}$.

By identifying $z_k \leftrightarrow \overbrace{x \cdots x}^{k-1} y$, we can associate $\mathbb{Q}\langle L_z\rangle$ with $\mathfrak{H}^1$. In what follows, we consider them to be equivalent. Using the usual power series multiplication, the linear map defined on the generators as

$$\begin{aligned} \zeta : \mathfrak{H}^0 &\longrightarrow \mathcal{Z}\,, \\ z_{k_1} \ldots z_{k_r} &\longmapsto \zeta(k_1, \ldots, k_r)\,. \end{aligned} \tag{1.4}$$

gives a $\mathbb{Q}$-algebra homomorphism $\zeta : (\mathfrak{H}^0, \bullet) \longrightarrow \mathcal{Z}$ for $\bullet \in \{*, \sqcup\!\sqcup\}$. That is, for any $w, v \in \mathfrak{H}^0$, $\zeta(w)\zeta(v) = \zeta(w \sqcup\!\sqcup v) = \zeta(w * v)$. Following the map ζ, elements of $\mathfrak{H}^1$ correspond the MZVs with arbitrary indices, while elements of $\mathfrak{H}^0$ correspond the absolutely convergence MZVs. By combining (1.3) and (1.4) over $\mathfrak{H}^0$, we can derive (1.1).

By the work of Hoffman ([Ho][HI]), any quasi-shuffle algebra can be equipped with the structure of a Hopf algebra using the deconcatenation coproduct Δ_{dec} which splits a word into different parts. For $w, u, v \in \mathfrak{H}^1$,

$$\Delta_{dec}(w) = \sum_{uv=w} u \otimes v.$$

For any Hopf algebra A with coproduct Δ and a $\mathbb{Q}$-algebra B with multiplication m and for $f, g \in \mathrm{Hom}(A, B)$, the convolution product is defined as

$$f \star g = m \circ (f \otimes g) \circ \Delta.$$

The antipode $S : A \to A$ is the inverse of Id with respect to $\star$. In quasi-shuffle algebra $(\mathfrak{H}^1, *, \Delta_{dec})$, the antipode S is given by

$$\sum_{i=0}^{r} S(z_{k_1} \ldots z_{k_i}) * z_{k_{i+1}} \ldots z_{k_r} = 0. \tag{1.5}$$

Acting map (1.4) on (1.5), we obtain

$$\zeta(S(z_{k_1} \ldots z_{k_r})) = (-1)^r \zeta^\star(k_r, \ldots, k_1).$$

Hence, the antipode formula for MZVs can be derived as follows:

Theorem 1.2 ([Ho])**.** *For $k_1, \dots, k_r \geq 2$, we have*

$$\sum_{i=0}^{r} (-1)^{r-i} \zeta(k_1, \dots, k_i) * \zeta^\star(k_r, \dots, k_{i+1}) = 0.$$

1.3 Schur MZV and multiple Schur function

Nakasuji, Phuksuwan and Yamasaki [NPY] introduced Schur multiple zeta values, which combinatorially interpolate both MZVs and MZSVs. It was first mentioned in [Ya] in the context of studying multiple Dirichlet L-values. We generalize their construction and introduce the notion of multiple Schur function as follows:

(i) A *partition* of a natural number n is a tuple $\lambda = (\lambda_1, \dots, \lambda_h)$ of positive integers $\lambda_1 \geq \cdots \geq \lambda_h \geq 1$ with $n = |\lambda| = \lambda_1 + \cdots + \lambda_h$. For another partition $\mu = (\mu_1, \dots, \mu_r)$ we write $\mu \subset \lambda$ if $r \leq h$ and $\mu_i \leq \lambda_i$ for $i = 1, \dots, r$, and we define *(skew) Young diagram* $D(\lambda/\mu)$ of λ/μ by

$$D(\lambda/\mu) = \left\{ (i,j) \in \mathbb{Z}^2 \mid 1 \leq i \leq h \,, \mu_i < j \leq \lambda_i \right\},$$

where $\mu_i = 0$ for $i > r$. In the case where $\mu = \varnothing$ is the empty partition (i.e., the unique partition of zero) we just write $\lambda/\mu = \lambda$. In the following we will just focus on the case $\mu = \varnothing$, but one should keep in mind that everything also makes sense for arbitrary μ. $C(\lambda) = \{(i,j) \in D(\lambda) | (i+1,j), (i,j+1) \notin D(\lambda)\}$ denote the set of corner of Young diagram $D(\lambda)$.

(ii) For an arbitrary set A and a partition λ denote by $\mathrm{YT}(\lambda, A)$ all *Young tableaux* with entries in A:

$$\mathrm{YT}(\lambda, A) = \{(m_{i,j})_{(i,j) \in D(\lambda)} \mid m_{i,j} \in A\}.$$

(iii) For a finite totally ordered set $\mathcal{X} = (X, \prec)$ define the *semi-standard Young tableaux* for a partition λ by

$$\mathrm{SSYT}(\lambda, \mathcal{X}) = \{(m_{i,j}) \in \mathrm{YT}(\lambda, X) \mid m_{i,j} \preceq m_{i,j+1}, m_{i,j} \prec m_{i+1,j}\}.$$

Here we write $a \preceq b$ if $a \prec b$ or $a = b$.

(iv) Let $\mathcal{R}$ be a commutative $\mathbb{Q}$-algebra, W an arbitrary set and $\mathcal{X}$ a finite totally ordered set. Then for a function

$$f : W \times \mathcal{X} \to \mathcal{R}$$

define the corresponding multiple Schur function for a Young tableaux $\Bbbk = (k_{i,j}) \in \mathrm{YT}(\lambda, W)$ by

$$S_f(\Bbbk) = \sum_{(m_{i,j}) \in \mathrm{SSYT}(\lambda, \mathcal{X})} \prod_{(i,j) \in D(\lambda)} f(k_{i,j}, m_{i,j}).$$

Example 1. *In the case $\mathcal{R} = \mathbb{R}$, $W = \mathbb{Z}$ and $\mathcal{X} = \{1, \ldots, M\}$ (with the usual order of natural numbers) for some $M \geq 1$ let*

$$\begin{aligned} f : \mathbb{Z} \times \{1, \ldots, M\} &\longrightarrow \mathbb{R} \\ (k, m) &\longmapsto \frac{1}{m^k}, \end{aligned}$$

With this setup, the above definition yields the (truncated) Schur multiple zeta values $\zeta_M(\Bbbk) = S_f(\Bbbk)$

By taking the limit as $\lim M \to \infty$, Schur multiple zeta values $\zeta(\Bbbk) = \lim_{M\to\infty} \zeta_M(\Bbbk)$ converges absolutely whenever $\Bbbk \in W_\lambda$, where

$$W_\lambda = \left\{ \Bbbk = (k_{ij}) \in \mathrm{SSYT}(\lambda, \mathbb{Z}) \;\middle|\; \begin{array}{l} k_{ij} \geq 1 \text{ for all } (i,j) \in D(\lambda) \setminus C(\lambda) \\ k_{ij} > 1 \text{ for all } (i,j) \in C(\lambda) \end{array} \right\}.$$

Schur multiple zeta values (Schur MZVs) preserve certain properties of the Schur function, such as Jacobi-Trudi formula.

Theorem 1.3 ([NPY] Jacobi-Trudi formula)**.** *Assume that $\Bbbk = (k_{ij}) \in \mathrm{SSYT}(\lambda/\mu,)$ and $k_{i,j} = a_{j-i}$ for all $(i,j) \in D(\lambda/\mu)$. $\lambda' = (\lambda'_1, \ldots, \lambda'_s) = \#\{j \mid \lambda_j \geq i\}$, $\mu' = (\mu'_1, \ldots, \mu'_s)$ are the conjugate of λ and μ. Then, we have*

$$\zeta_{\lambda/\mu}(\Bbbk) = \det \left[\zeta(a_{-\mu'_j+j-1}, a_{-\mu'_j+j-2}, \ldots, a_{-\mu'_j+j-(\lambda'_i-\mu'_j-i+j)}) \right]_{1\leq i,j\leq s}. \tag{1.6}$$

Here, we understand that $\zeta(\cdots) = 1$ if $\lambda'_i - \mu'_j - i + j = 0$ and 0 if $\lambda'_i - \mu'_j - i + j < 0$.

2 Main Results

Our main results are presented in two parts. In the first part, we introduce some analytic properties of Schur multiple Eisenstein series, including its precise definition, Fourier expansion, Jacobi-Trudi formula, and modularity theory. In the second part, we explore the algebraic structure of Schur multiple Eisenstein series by connecting them to the $\mathbb{Q}$-vector space of multiple zeta values. This result is then extended to the algebraic structure of Young tableaux.

2.1 Schur Multiple Eisenstein series

To define Schur multiple Eisenstein series, we set $\mathbb{Z}_M = \{-M, \ldots, -1, 0, 1, \ldots, M\}$ and consider, for $M, N \geq 1$ and $\tau \in \mathbb{H}$ the set $X^\tau_{M,N} = \mathbb{Z}_M\tau + \mathbb{Z}_N$. This gives a finite ordered set $\mathcal{X}^\tau_{M,N} = (X^\tau_{M,N}, \prec)$. Next, we consider the subset of "positive" lattice points defined as

$$X^{\tau,>0}_{M,N} = \{\lambda \in X^\tau_{M,N} \mid 0 \prec \lambda\}.$$

This gives another finite ordered set $\mathcal{X}^{\tau,>0}_{M,N} = (X^{\tau,>0}_{M,N}, \prec)$ We then define the map

$$\begin{aligned} f : \mathbb{Z} \times X^{\tau,>0}_{M,N} &\to \mathbb{R} \\ (k, m\tau + n) &\longmapsto \frac{1}{(m\tau + n)^k}, \end{aligned}$$

and introduce the ***(truncated) Schur multiple Eisenstein series*** for $\Bbbk \in \mathrm{YT}(\mathbb{Z})$ by $G_{M,N}(\Bbbk) = S_f$. More explicitly, for a partition λ and $\Bbbk = (k_{i,j}) \in \mathrm{YT}(\lambda, \mathbb{Z})$, these are defined as

$$G_{M,N}(\Bbbk;\tau) = \sum_{(m_{i,j}\tau+n_{i,j})\in \mathrm{SSYT}(\lambda,\mathcal{X}^{\tau,>0}_{M,N})} \prod_{(i,j)\in D(\lambda)} \frac{1}{(m_{i,j}\tau + n_{i,j})^{k_{i,j}}}\,. \tag{2.1}$$

We can now establish the following proposition:

Proposition 2.1. *If $k_{i,j} \geq 2$ for all $(i,j) \in D(\lambda)$ then the following limit exists*

$$G(\Bbbk;\tau) := \lim_{M\to\infty} \lim_{N\to\infty} G_{M,N}(\Bbbk;\tau)\,. \tag{2.2}$$

This is the definition of the ***Schur multiple Eisenstein series***. If we defined Schur multiple Eisenstein series without using the limit of summation, the conditions $k_{i,j} \geq 3\,((i,j) \in C(\lambda/\mu))$, $k_{i,j} \geq 2\,((i,j) \in D(\lambda/\mu) \setminus C(\lambda/\mu))$ are necessary for the absolute convergence of the sum. However, by a similar argument as in [BT], the order of the limits in (2.2) ensures convergence even in the case where $k_{i,j} = 2$ for $(i,j) \in C(\lambda/\mu)$. This approach aligns with the usual method for defining the quasi-modular form $G(2,\tau)$. The construction of the Fourier expansion described below uses exactly this Eisenstein summation:

Theorem 2.2 ([Yu], Fourier expansion). *For $\Bbbk = (k_{i,j};\lambda/\mu) \in \mathrm{YT}$, and $q = e^{2\pi i\tau}$, there exist explicit $\alpha^{\Bbbk}_{\mathbb{l},\mathbb{h}} \in \mathbb{Z}$ such that*

$$G(\Bbbk) = \zeta(\Bbbk) + \sum_{|\Bbbk|=|\mathbb{l}|+|\mathbb{h}|} \alpha^{\Bbbk}_{\mathbb{l},\mathbb{h}}\zeta(\mathbb{l})g(\mathbb{h}) + g(\Bbbk). \tag{2.3}$$

Also, we have

$$G(\Bbbk) = \zeta(\Bbbk) + \sum_{\mu\subsetneq\eta\subsetneq\lambda} \zeta(\Bbbk_{\eta/\mu})\check{g}(\Bbbk_{\lambda/\eta}) + \check{g}(\Bbbk). \tag{2.4}$$

In particular,

$$G(\Bbbk;\tau) = \zeta(\Bbbk) + \sum_{n>0} a_{\Bbbk}(n)q^n \quad \text{for} \quad a_{\Bbbk}(n) \in \mathcal{Z}[\pi i].$$

Here we show two ways to writing the Fourier expansion of th Schur multiple Eisenstein series. Equation (2.3), analog of theorem1.1, represents the classical form of the Fourier expansion of Eisenstein series. It extends multiple q-series $g(k_1,\ldots,k_r)$ in (1.2) to the Schur versions, given by

$$g(\Bbbk_{\lambda/\mu}) = g(\Bbbk_{\lambda/\mu};\tau) = \sum_{m_{i,j}\in \mathrm{SSYT}(\lambda/\mu)} \prod_{(i,j)\in D(\lambda/\mu)} \Psi(k_{i,j}; m_{i,j}\tau) \in \mathbb{Q}[\pi i][\![q]\!].$$

On the other hand, in the summation of (2.1), the terms can be split into two parts based on whether $m_{i,j} = 0$ or $m_{i,j} \geq 1$. The part where $m_{i,j} = 0$ is in fact equal a Schur multiple zeta values. We denote the summation where $m_{i,j} \geq 1$ as $\check{g}$.

Using the semi-standard condition, the Young tableau can be divided into two parts: the top-left part corresponds to ζ, and the bottom-right part corresponds to $\check{g}$. This decomposition gives rise to (2.4), where the summation in the red region represents Schur MZVs, and the summation in the blue region corresponds to $\check{g}$ in the following figure.

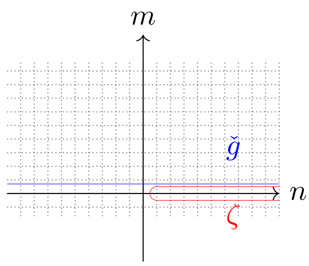

Figure 1: The region corresponding to the summation of ζ and $\breve{g}$

Example 2. *For example, the Fourier expansion of* $G\left(\begin{ytableau} 2 & 2 \\ 2 \end{ytableau}\right)$ *can be written as:*

$$\begin{aligned}
G\left(\begin{ytableau} 2 & 2 \\ 2 \end{ytableau}\right) &= \zeta\left(\begin{ytableau} 2 & 2 \\ 2 \end{ytableau}\right) + \zeta\left(\begin{ytableau} 2 \\ 2 \end{ytableau}\right)\breve{g}\left(\begin{ytableau} 2 \end{ytableau}\right) + \zeta\left(\begin{ytableau} 2 & 2 \end{ytableau}\right)\breve{g}\left(\begin{ytableau} 2 \end{ytableau}\right) + \zeta\left(\begin{ytableau} 2 \end{ytableau}\right)\breve{g}\left(\begin{ytableau} \none & 2 \\ 2 \end{ytableau}\right) + \breve{g}\left(\begin{ytableau} 2 & 2 \\ 2 \end{ytableau}\right) \\
&= \zeta\left(\begin{ytableau} 2 & 2 \\ 2 \end{ytableau}\right) + \zeta\left(\begin{ytableau} 2 \\ 2 \end{ytableau}\right) g\left(\begin{ytableau} 2 \end{ytableau}\right) + \zeta\left(\begin{ytableau} 2 & 2 \end{ytableau}\right) g\left(\begin{ytableau} 2 \end{ytableau}\right) + \zeta(2)g(4) + 4\zeta(2)g(2,2) + g\left(\begin{ytableau} 2 & 2 \\ 2 \end{ytableau}\right).
\end{aligned}$$

Proposition 2.3 ([Yu])**.** *The* $\breve{g}$ *can be written as a MZV-linear combination of* g*.*

Example 3.

$$\breve{g}\left(\begin{ytableau} 2 & 2 \\ 2 \end{ytableau}\right) = g\left(\begin{ytableau} 2 & 2 \\ 2 \end{ytableau}\right) + 2\zeta(2)g\left(\begin{ytableau} 2 \\ 2 \end{ytableau}\right).$$

Let's consider a special Young tableau, where all the diagonal variables are the same. In this case, we can generalize many formulas for Schur polynomials to Schur multiple Eisenstein series, such as Jacobi-Trudi formula. Following the similar argument of [NPY], we obtain the following result:

Theorem 2.4 ([Yu], Jacobi-Trudi formula)**.** *Schur multiple Eisenstein series also satisfy the Jacobi-Trudi formula.*

Example 4. *For example, when* $\lambda/\mu = (4,3,2)/(2,1)$*, with* $a,b,c,d,e,f \in \mathbb{N}_{\geq 2}$*, we have*

$$G\left(\begin{ytableau} \none & d & e & f \\ \none & c & d \\ a & b \end{ytableau}\ ;\tau\right) = \det\begin{pmatrix} G(a;\tau) & G(d,c,b,a;\tau) & G(e,d,c,b,a;\tau) & G(f,e,d,c,b,a;\tau) \\ 1 & G(d,c,b;\tau) & G(e,d,c,b;\tau) & G(f,e,d,c,b;\tau) \\ 0 & G(d;\tau) & G(e,d;\tau) & G(f,e,d;\tau) \\ 0 & 0 & 1 & G(f;\tau) \end{pmatrix}.$$

Lemma 2.5 ([Yu])**.** *For* $k \geq 2$*, we have*

$$\exp\left(\sum_{i\geq 1}\frac{(-1)^{i-1}}{i}G(ik;\tau)X^i\right) = \sum_{n=0}^{\infty} G(\underbrace{k,\ldots,k}_{n};\tau)X^n.$$

Lemma 2.5 is an application of Hoffman-Ihara's [HI] result. It allows us to rewrite the multiple Eisenstein series with the index that all variables are k, as a polynomial in the Eisenstein series with index $k \times l$ for $l \in \mathbb{N}$. By choosing a special case where Young tableau with all variables are same, Corollary 2.4 tell us that these Schur multiple Eisenstein series can be written as a polynomial of multiple Eisenstein series with same variables. By lemma 2.5, it can also be written as a polynomial of Eisenstein series with multiple indices. On the other hand, we have the following relations:

$$\begin{aligned} G(2;\tau) &= G\left(\,\boxed{2}\,;\tau\right), \ G(4;\tau) = G\left(\,\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array}\,;\tau\right) - G\left(\begin{array}{|c|}\hline 2 \\ \hline 2 \\ \hline \end{array}\,;\tau\right), \\ G(6;\tau) &= G\left(\,\begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline \end{array}\,;\tau\right) - G\left(\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \end{array};\tau\right) + G\left(\begin{array}{|c|}\hline 2 \\ \hline 2 \\ \hline 2 \\ \hline \end{array}\,;\tau\right). \end{aligned}$$

Following these formulas, we can then obtain the modularity results.

Corollary 2.6 ([Yu], Modularity). *Let $\Bbbk \in \mathrm{YT}$ with $k_{i,j} = k \geq 2$, then*

(i) $G(\Bbbk;\tau) \in \mathbb{Q}[G(kl;\tau) | l \geq 1]$.

*(ii) If $k = 2$, $G(\Bbbk;\tau)$ is quasi modular form, and **every** quasi modular forms can be written as a linear combination of these $G(\Bbbk;\tau)$.*

(iii) For even $k \geq 4$, $G(\Bbbk;\tau)$ is a modular form.

2.2 Algebra structure of Young tableaux

Let $\mathbb{Q}\,\mathrm{YT}$ be the $\mathbb{Q}$-vector space generated by the set YT. Let $D_1, \ldots D_r$ be non-empty subsets of the skew Young diagram $D(\lambda/\mu)$ that provide a disjoint decomposition of $D(\lambda/\mu)$, i.e., $D(\lambda/\mu) = D_1 \sqcup \cdots \sqcup D_r$. A tuple $(D_1, \ldots, D_r)$ with $D_a \subset D(\lambda/\mu)$ is called semi standard decomposition of λ/μ if

(i) $D(\lambda/\mu) = D_1 \sqcup \cdots \sqcup D_r$,

(ii) the Young tableau $(t_{ij}) \in YT(\lambda/\mu)$ with $t_{ij} = a$ if $(i,j) \in D_a (a = 1, \ldots, r)$ is semi standard.

Let $SSD(\lambda/\mu)$ denote the set of all semi-standard decompositions of $D(\lambda/\mu)$. For example,

$$SSD((2,1)) = SSD\left(\begin{array}{|c|c|}\hline \ & \ \\ \hline \multicolumn{1}{|c|}{\ } \\ \cline{1-1} \end{array}\right) = \left\{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array}\right\},$$

where in the case $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \end{array}$, $D_1 = \{(1,1),(2,1)\} = \begin{array}{|c|c|}\hline \ & \ \\ \hline \end{array}$, $D_2 = \{(2,1)\} = \boxed{\ }$.

Define the linear map:

$$\begin{aligned} L : \mathbb{Q}\,\mathrm{YT} &\longrightarrow \mathfrak{H}^1_* \\ \Bbbk = (k_{i,j}) &\longmapsto \sum_{((D_1,\ldots,D_r) \in SSD(\lambda/\mu))} z_{\sum_{(i,j)\in D_1} k_{i,j}} \cdots z_{\sum_{(i,j)\in D_r} k_{i,j}}. \end{aligned}$$

Let the space $\mathcal{Y}$ be defined as:

$$\mathcal{Y} = {}^{\mathbb{Q}\,\mathrm{YT}}\!\big/_{\ker(L)}.$$

By definition, we have $\mathcal{Y} \cong \mathfrak{H}^1$ as $\mathbb{Q}-$vector space.

We can express the product of two Schur multiple Eisenstein series as a single Schur multiple Eisenstein series by placing one Young tableau on the top right of the other:

$$G\left(\begin{ytableau} \none & a & b \\ c & d \\ e \\ f \end{ytableau}\,;\tau\right) G\left(\begin{ytableau} g & h \\ i \end{ytableau}\,;\tau\right) = G\left(\begin{ytableau} \none & \none & \none & g & h \\ \none & \none & \none & i \\ \none & a & b \\ c & d \\ e \\ f \end{ytableau}\,;\tau\right).$$

Following this idea, we define the harmonic product $\check{*}$ of Young tableaux by attaching one to the top right of the other. For tuples $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\mu = (\mu_1, \ldots, \mu_n)$, $\lambda' = (\lambda'_1 \ldots, \lambda'_m)$, $\mu' = (\mu'_1, \ldots, \mu'_m)$ and Young tableaux $\mathbb{k} = \mathbb{k}_{\lambda/\mu}$, $\mathbb{h} = \mathbb{h}_{\lambda'/\mu'} \in \mathrm{YT}$, we define:

$$\mathbb{k}\check{*}\mathbb{h} = \mathbb{l}_{\alpha/\beta},$$

where the Young diagram α/β is given by $\alpha = (\lambda'_1 + \lambda_1, \lambda'_1 + \lambda_2, \ldots, \lambda'_1 + \lambda_n, \lambda'_1, \ldots, \lambda'_m)$ and $\beta = (\mu_1 + \lambda'_1, \ldots, \mu_n - \lambda'_1, \mu'_1, \ldots, \mu'_m)$.

Theorem 2.7 ([Yu])**.** *$(\mathcal{Y}, \check{*})$ is a algebra.*

Proposition 2.8 ([Yu])**.** *For $\mathbb{k}$, $\mathbb{h} \in \mathrm{YT}$, we have:*

$$L(\mathbb{k}\check{*}\mathbb{h}) = L(\mathbb{k}) * L(\mathbb{h}).$$

This proposition shows L is a bijective algebra morphism from $(\mathcal{Y}, \check{*})$ to $(\mathfrak{H}^1, *)$.

On the other hand, following the Fourier expansion (2.4), Schur multiple Eisenstein series can be written as a decomposition of Schur multiple zeta values and Schur type $\breve{g}$:

$$G(\mathbb{k}_{\lambda/\mu}; \tau) = \sum_{\mu \subseteq \eta \subseteq \lambda} \zeta(\mathbb{k}_{\eta/\mu}) \breve{g}(\mathbb{k}_{\lambda/\eta}).$$

Forcing on the index, we can generalize this decomposition to the $\mathbb{Q}\,\mathrm{YT}$ as the deconcatenation coproduct $\check{\Delta}$,

$$\begin{aligned} \check{\Delta} : \mathbb{Q}\,\mathrm{YT} &\longrightarrow \mathbb{Q}\,\mathrm{YT} \otimes \mathbb{Q}\,\mathrm{YT} \\ \mathbb{k}_{\lambda/\mu} &\longmapsto \sum_{\mu \subseteq \eta \subseteq \lambda} \mathbb{k}_{\eta/\mu} \otimes \mathbb{k}_{\lambda/\eta}. \end{aligned}$$

Proposition 2.9. *For $\mathbb{k} \in \mathrm{YT}$, we have*

$$L(\check{\Delta}(\mathbb{k})) = \Delta(L(\mathbb{k})).$$

Note that both the harmonic product and the deconcatenation coproduct are graded by the number of boxes in the Young tableaux. From this, we obatin the following results:

Theorem 2.10 ([Yu])**.** *(i) $(\mathcal{Y}, \check{\Delta})$ is a coalgebra, which is isomorphic to $(\mathfrak{H}^1, \Delta_{dec})$*

(ii) $(\mathcal{Y}, \check{*}, \check{\Delta})$ is a Hopf-algebra, which is isomorphic to $(\mathfrak{H}^1, *, \Delta_{dec})$.

Further, without considering the linear map L, we can also show the following:

Theorem 2.11 ([Yu])**.** *$(\mathbb{Q}\,\mathrm{YT}, \check{*}, \check{\Delta})$ is a graded-Hopf algebra.*

This theorem implies that for any symmetric multiple Schur function F, we can find two symmetric multiple Schur function U and V such that $F = \sum U \otimes V$, and F satisfy the stuffle relation.

3 Future work

Future work will be carried out on two fronts in the short term. The first area of focus is the refinement of Theorem 2.10 and 2.11. One of the most important features of being a Hopf algebra is that it consists of an antipode S. While we can technically write the antipode on the Hopf-algebra $\mathcal{Y}$ using the inverse of the map L on the antipode in $\mathfrak{H}^1_*$. But it does not relate directly to a transformation of Young tableaux. Our goal is to represent the antipode by cutting and inverting the Young tableaux, and this representation should be suitable for both $\mathbb{Q}\,\mathrm{YT}$ and $\mathcal{Y}$.

The second area of focus is generalized the shuffle product $ш$ to Schur multiple Eisenstein series and $\mathbb{Q}\,\mathrm{YT}$. Hirose[HMO] provides a Yamamoto integral representation of the Schur multiple zeta values with same diagonal variables. Although, for the moment we cannot find an exact iterated integral representation of Eisenstein series. Bachmann and Tasaka [BT] discovered an explicit connection between the Fourier expansion of multiple Eisenstein series and the Goncharov coproduct on Hopf algebras of iterated integrals. We hope to follow these ideas to build a connection between $(\mathbb{Q}\,\mathrm{YT}, \check{ш}, \check{\Delta}_{Gon})$ and Schur multiple Eisenstein series. Our task will be to define $\check{ш}$ and $\check{\Delta}_{Gon}$ in a precise manner and to study the algebra structure of $(\mathbb{Q}\,\mathrm{YT}, \check{ш}, \check{\Delta}_{Gon})$.

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